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LETTER TO THE EDITOR

Exact solution of the N-dimensional generalized Dirac–Coulomb equation

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Abstract. An exact solution to the bound state problem for the N-dimensional generalized Dirac-Coulomb equation, whose potential contains both the Lorentz-vector and Lorentz-scalar terms of the Coulomb form, is obtained

Recently, various $(1/N)$-expansion procedures, where N is the number of spatial dimensions, have been suggested to obtain analytical expressions for Dirac eigenvalues and eigenfunctions (Miramontes and Pajares 1984, Roychoudhuri and Varshni 1987, 1988, 1989, Panja and Dutt 1988, Atag 1989, Roy and Roychoudhuri 1990, Panja *et al* 1990, Stepanov and Tutik 1991). In common practice an opportunity to examine the applicability and accuracy of one or another approximation scheme is mainly provided with the solvable examples. As regards the Dirac equation, the exact solutions are available only for a few special cases (Berestetskiĭ *et al* 1971, Bagrov *et al* 1982). In N space dimensions the scope of such examples is, in essence, limited to the Coulomb potential which is considered as the time component of a Lorentz four-vector (Joseph 1967, Coulson and Joseph 1967, Wong 1990). However, a Lorentz-scalar interaction is of great importance in the context of the relativistic quark model. It is employed for describing magnetic moments (Bogoliubov *et al* 1965, Lipkin and Tavkhelidze 1965, Bogoliubov 1972) and avoiding the Klein paradox risen from the quarkonium confining potentials (Critchfield 1975, Bunion and Li 1975). Therefore the search for exact solutions to problems concerned with the scalar interaction takes on special significance.

The purpose of this letter is to show that the N-dimensional generalized Dirac–Coulomb equation, involving a Coulomb potential in the form of a superposition of the Lorentz-vector and Lorentz-scalar terms, has an exact solution.

1. *Radial equations.* We study the bound state problem for a single fermion moving in an attractive central potential. This potential contains both the time component of a Lorentz four-vector, $V_v(r) = -b/r$, just like the Coulomb potential of the hydrogenic atom, and the Lorentz-scalar term, $V_s(r) = -a/r$. Then the N-dimensional generalized Dirac-Coulomb equation we wish to solve is

$$\left(c\rho_1 \sum_{i=1}^{N} p_i\sigma_{i,N+1} + \rho_3\left(mc^2 - \frac{a}{r}\right) - \frac{b}{r}\right)\Psi = e\Psi \qquad (1)$$

where the notation of Joseph (1967), Coulson and Joseph (1967) is used.

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Accepting the next representation of the eigenfunction Ψ_{Ejlm}:

$$\Psi_{Ejlm}=\begin{pmatrix} g(r) & \Omega_{jlm} \\ if(r) & \Omega_{jl+1\,m}\end{pmatrix} \tag{2}$$

for the radial equations we have (Coulson and Joseph 1967)

$$\begin{aligned}&\frac{dg}{dr}+\frac{\chi+(N-1)/2}{r}g-\frac{1}{\hbar c}\left(E+mc^2-\frac{a-b}{r}\right)f=0\\ &\frac{df}{dr}+\frac{\chi-(N-1)/2}{r}f+\frac{1}{\hbar c}\left(E-mc^2+\frac{a+b}{r}\right)g=0\end{aligned} \tag{3}$$

where $\chi=s(N+2j-2)/2$ is the invariant of the N-dimensional Dirac equation, with $j=l-s/2$ and $s=\pm1$ denoted the sign of χ. In what follows, we shall put $\hbar=c=1$ and consider only positive values of the energy, i.e. the 'particle' case. We intend to obtain the solutions to equation (3) with the standard procedure described in (Berestetskiĭ *et al* 1971).

Let us change r to the dimensionless variable ρ with $\rho=2r(m^2-E^2)^{1/2}=2\mu r$. To incorporate the regular behaviour of the radial functions at the origin we write

$$\left.\begin{matrix} g(\rho) \\ f(\rho)\end{matrix}\right\}=\pm\sqrt{1\pm\frac{E}{m}}\,e^{-\rho/2}\rho^{\gamma-(N-1)/2}(Q_1(\rho)\pm Q_2(\rho)) \tag{4}$$

where $\gamma=(\chi^2+a^2-b^2)^{1/2}\geqslant 0$.

The system (3) is transformed then into

$$\begin{aligned}&\rho Q_1'+\left(\gamma-\frac{am+bE}{\mu}\right)Q_1+\left(\chi-\frac{aE+bm}{\mu}\right)Q_2=0\\ &\rho Q_2'+\left(\frac{\gamma+am-bE}{\mu}-\rho\right)q_2+\left(\chi+\frac{aE+bm}{\mu}\right)Q_1=0.\end{aligned} \tag{5}$$

Eliminating Q_2 and Q_1, respectively, leads to our basic equations

$$\begin{aligned}&\rho Q_1''+(2\gamma+1-\rho)Q_1'-\left(\gamma-\frac{am+bE}{\mu}\right)Q_1=0\\ &\rho Q_2''+(2\gamma+1-\rho)Q_2'-\left(\gamma+1-\frac{am+bE}{\mu}\right)Q_2=0\end{aligned} \tag{6}$$

which have the identical form with the equation for the hypergeometric confluent function ${}_1F_1(\alpha,\beta,\rho)$ (Buchholz 1953)

$$\rho F''+(\beta-\rho)F'-\alpha F=0. \tag{7}$$

Therefore, we obtain

$$\begin{aligned}&Q_2=A\,{}_1F_1\left(\gamma-\frac{am+bE}{\mu},2\gamma+1,\rho\right)\\ &Q_2=B\,{}_1F_1\left(\gamma+1-\frac{am+bE}{\mu},2\gamma+1,\rho\right).\end{aligned} \tag{8}$$

However the coefficients A and B are not independent. Considering equations (5) at $\rho=0$ and making use of the property: ${}_1F_1(\alpha,\beta,0)=1$, it is observed that

$$B=-A\frac{\gamma-(am+bE)/\mu}{\chi-(aE+bm)/\mu}. \tag{9}$$

2. *Energy eigenvalues.* For the confluent hypergeometric function ${}_1F_1(\alpha, \beta, \rho)$ to terminate, its first argument α should be a negative integer. Hence we must put

$$\gamma - \frac{am + bE}{\mu} = -n \qquad n = 0, 1, 2, \ldots . \tag{10}$$

Omitting the standard discussion (Berestetskiĭ *et al* 1971), we just present the final result. Thus, the energy spectrum, we have obtrained for the bound states of the N-dimensional generalized Dirac–Coulomb equation, is written (with $\hbar$ and c restored) as

$$E = \frac{mc^2}{(\gamma+\tilde{n})^2 + (b/\hbar c)^2}[(\gamma+\tilde{n})\sqrt{(\gamma+\tilde{n})^2 - (a/\hbar c)^2 + (b/\hbar c)^2} - ab/(\hbar c)^2] \tag{11}$$

where

$$\tilde{n} = n + (s+1)/2 = \begin{cases} 0, 1, 2, \ldots & s = -1 \\ 1, 2, 3, \ldots & s = 1 \end{cases}$$

and n denotes the radial quantum number.

For the pure vector potential ($a = 0$, $b \neq 0$), this reproduces the known expression (Coulson and Joseph 1967, Wong 1990, Stepanov and Tutik 1991)

$$E = mc^2\left[1 + \frac{(b/\hbar c)^2}{(\tilde{n} + \sqrt{\chi^2 - (b/\hbar c)^{2)2}}}\right]^{-1/2}. \tag{12}$$

Retaining only the scalar term ($a \neq 0$, $b = 0$), we find

$$E = mc^2\left[1 - \frac{(a/\hbar c)^2}{(\tilde{n} + \sqrt{\chi^2 + (a/\hbar c)^2})^2}\right]^{1/2}. \tag{13}$$

From equation (13) it is seen that, contrary to the pure vector case, the pure scalar potential may be of arbitrary high strength. For a superposition of vector and scalar potentials, the square root in the definition of the quantity γ becomes imaginary unless $\chi > b^2 - a^2$, causing the breakdown of the bound-state solution.

Note too that the presence of the scalar term in the potential does not remove the energy spectrum degeneracy inherent in the hydrogen atom.

3. *Radial eigenfunctions.* It is known (Buchholz 1953), that that the confluent hypergeometric function is connected with the generalized Laguerre polynomial as follows:

$${}_1F_1(-n, \alpha+1, \rho) = \frac{\Gamma(\alpha+1)n!}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(\rho). \tag{14}$$

Hence, the radial solution we have derived can be expressed as

$$\left.\begin{matrix} g(\rho) \\ f(\rho) \end{matrix}\right\} = \pm A\sqrt{1 \pm \frac{E}{m}}\, \mathrm{e}^{-\rho/2}\rho^{\gamma-(N-1)/2}\frac{\Gamma(2\gamma+1)n!}{\Gamma(2\gamma+n+1)}$$

$$\times\left[L_n^{(2\gamma)}(\rho) \mp \frac{1}{n}\left(\chi + \frac{aE+bm}{\mu}\right)L_{n-1}^{(2\gamma)}(\rho)\right] \tag{15}$$

where for convenience of notation we use n for $\tilde{n}$.

Let us obtain the normalization constant. The angular parts of the wavefunctions are already normalized. The radial parts are normalized according to the condition

$$\int_0^\infty (g^2(r)+f^2(r))r^{N-1}\,\mathrm{d}r=1 \tag{16}$$

which in terms of the dimensionless variable $\rho=2\mu r$ becomes

$$\int_0^\infty (g^2(\rho)+f^2(\rho))\rho^{N-1}\,\mathrm{d}\rho=(2\mu)^N. \tag{17}$$

From any of the standard textbooks on special functions (see for instance Tutik 1983) we have

$$\int_0^\infty \mathrm{e}^{-\rho}\rho^{2\gamma}L_n^{(2\gamma)}(\rho)L_{n'}^{(2\gamma)}(\rho)\,\mathrm{d}\rho=\frac{\Gamma(2\gamma+n+1)}{n!}\delta_{nn'}. \tag{18}$$

Therefore the normalization constant A should read

$$A=\frac{(2\mu)^{N/2}}{\Gamma(2\gamma+1)}\sqrt{\frac{(aE+bm-\mu\chi)\Gamma(2\gamma+n+1)}{4(aE+bm)n!}}. \tag{19}$$

Finally, our solution is

$$\left.\begin{matrix} g(r) \\ f(r)\end{matrix}\right\}=\pm(2\mu)^{\gamma+1/2}\sqrt{\frac{(1\pm E/m)(aE+bm-\mu\chi)n!}{4(aE+bm)\Gamma(2\gamma+n+1)}}$$
$$\times r^{\gamma-(N-1)/2}\,\mathrm{e}^{-\mu r}\left[L_n^{(2\gamma)}(2\mu r)\mp\frac{1}{n}\left(\chi+\frac{aE+bm}{\mu}\right)L_{n-1}^{(2\gamma)}(2\mu r)\right]. \tag{20}$$

Notice that in the case of the pure vector potential in three dimensions, this expression coincides with the known solution to the Dirac–Coulomb problem (Berestetskiĭ *et al* 1971).

To summarize, we obtain an exact solution of the N-dimensional generalized Dirac–Coulomb equation whose potential consists of the Lorentz-scalar and Lorentz-vector terms. Aside from the theoretical significance, this result will favour for better understanding the contribution of the Lüscher term appeared in the string model potential (Lüscher 1981, Fishbane *et al* 1986, Braaten and Tse 1987, German and Kleinert 1989), which will be published elsewhere.

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